

de Sitter–holographic C functions with a topological, dilatonic twist

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Recently, the holographic aspects of asymptotically de Sitter spacetimes have generated substantial literary interest. The plot continues in this paper, as we investigate a certain class of dilatonic deformed “topological” de Sitter solutions [which were introduced by R. Cai, Y. Myung, and Y. Zhang, *Phys. Rev. D* **65**, 084019 (2002)]. Although such solutions possess a detrimental cosmological singularity, their interpretation from a holographic perspective remains somewhat unclear. The current focus is on the associated generalized C functions, which are shown to maintain their usual monotonicity properties in spite of this exotic framework. These findings suggest that such topological solutions may still play a role in our understanding of quantum gravity with a positive cosmological constant.

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I. INTRODUCTION

There is considerable observational evidence that the physical universe has a positive (albeit, disturbingly small) cosmological constant [1]. This observation is at least partially responsible for the recent flurry of investigations into asymptotically de Sitter spacetimes. In particular, the various holographic aspects of de Sitter (dS) space have garnered much attention. (See Ref. [2] for a list of relevant citations. Also see Refs. [3–13] for more recent work.)

The focal point of dS-based holography has been a conjectured duality between asymptotically dS spacetimes and conformal field theories (CFTs) [14]. This dS-CFT duality can be viewed as an analogy to the celebrated anti-de Sitter (AdS)-CFT correspondence [15–17] (which, in turn, is an explicit realization of the renowned holographic principle [18,19]). The CFT in a dS-inspired duality, just like its AdS analog, lives on an asymptotic boundary of the bulk spacetime. However, contrary to the AdS case, the dS boundary is spacelike (located at temporal infinity) and the dual CFT is necessarily a Euclidean one. These distinctions can be attributed to the absence of both a globally timelike Killing vector and a spatial infinity in asymptotically dS spacetimes [20].

The various investigations into dS holography have, of course, lead to many interesting deductions and observations. At the forefront of these is Bousso’s realization of an entropic upper bound [21]. More specifically, the entropy of pure dS space serves as an upper bound on the total entropy that can be stored in any spacetime with a positive cosmological constant.¹ With guidance from the “Bousso bound,” Balasubramanian, de Boer and Minic (BdBm) [23] have proposed a similar upper limit on the total mass of an asymptotically dS spacetime. In particular, these authors have conjectured that any such spacetime whose conserved mass²

exceeds that of pure dS space will contain a naked cosmological singularity.

For the sake of argument, let us accept the conjectured mass bound as being a true property of asymptotically dS spacetimes (as recent analysis does seem to support [26]). In this case, from a bulk viewpoint, the implications are quite severe; a violation of this bound results in a type of singularity that is considered to be *non grata* in most cosmological models. On the other hand, from a holographic perspective, the implications are somewhat murkier. As pointed out by Ghezalbash and Mann [27], for a hypothetical observer located on an asymptotic spacelike boundary (i.e., a “CFT observer”), any such singularity will remain causally hidden behind the cosmological horizon. That is to say, any quantity measured by this observer depends only on the boundary theory; in fact, the observer need not be aware that an interior region even exists. To reword this in a philosophical sense, what exactly constitutes the “true physical” picture: the boundary theory or the bulk (or both or neither)?

Given the stated ambiguity, one might argue that bound-violating asymptotically dS solutions should not be dismissed *a priori*. Such solutions have been explicitly formulated in a paper by Cai, Myung, and Zhang [26] (with their original motivation being to test the mass-bound conjecture). One of these so-called “topological” de Sitter (TdS) solutions can effectively be obtained with a sign reversal (in the mass term) of a more conventional Schwarzschild-dS solution. As a consequence, the black hole horizon disappears, leaving behind a naked singularity enclosed by the usual cosmological horizon.³ Furthermore, it can readily be confirmed that, for any TdS solution, the conserved mass (in accordance with the definition of Ref. [23]) does indeed exceed that of its purely dS counterpart.

Since the original presentation by Cai *et al.* [26], some subsequent papers have considered the implications of a pos-

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¹Note that the validity of the Bousso bound does depend on some form of the positive energy condition [22].²The “BdBm bound” implies a specific definition for the mass [23,24]. This definition can be viewed as an appropriate generalization of the Brown-York quasi-local energy [25].³It should be kept in mind that a TdS solution can have a spherical, hyperbolic or flat (cosmological) horizon geometry. This is contrary to the Schwarzschild-dS case, which only permits a spherical horizon geometry. It is this diversity in TdS horizon geometries that prompted the topological nomenclature in analogy with Ref. [28].

sible TdS/CFT duality [29–31,2]. For the most part, this duality would appear to be preferential to its Schwarzschild-dS counterpart,⁴ inasmuch as the CFT energy can only be positive in the TdS case [32,29]. (Significantly, a negative energy implies a non-unitary theory.) In view of this desirable feature, we argue that TdS solutions merit further investigation, and proceed on this basis.

Ultimately, one might hope that quantum gravity can be used to deduce the validity (or invalidity) of a field theory that is holographically dual to a TdS bulk. However, as it is well known, a definitive theory of quantum gravity remains currently out of reach. Even the most promising candidate, string theory, fails to provide a suitable description of dS space [20]. Nevertheless, we propose that much can still be learned by subjecting the (conjectured) TdS-CFT correspondence to various holographic “consistency checks.”

With the above proposal in mind, let us consider the intriguing phenomena of holographic renormalization group (RG) flows. Significantly, RG flows are commonly believed to play a prominent role in any holographic bulk-boundary duality. (For instance, see Refs. [33–37,23].) More specifically, as any relevant parameter of a bulk spacetime evolves, a RG flow is expected to be induced in the dually related boundary theory. This picture follows from the so-called ultraviolet-infrared correspondence [38,39], which implies that bulk evolution towards the infrared translates into boundary flow towards the ultraviolet and *vice versa*.

What is particularly pertinent to the holographic-RG picture is the existence of a generalized C function (in analogy with RG flows in a two-dimensional CFT context [40]). Moreover, this C function should exhibit various monotonicity properties that are reflective of the underlying ultraviolet-infrared duality. In view of all this, an appropriate analysis of a prospective C -function should serve as a suitable “laboratory” for testing the viability of any conjectured bulk-boundary correspondence. Just such an analysis, in a TdS context, will serve as the focal point of the current paper.

If one sets out to “test” a prospective C function, it should be significantly more informative when non-trivial matter fields are allowed in the bulk theory. (See, for instance, Ref. [41].) For this reason, we will generalize our considerations to a certain class of solutions that can be viewed as dilatonic deformations of a TdS geometry [26]. This new class can alternatively be viewed as domain wall solutions having a (flat) cosmological horizon. In fact, these solutions are essentially analytic continuations of domain wall-black hole spacetimes that effectively describe a truncated theory of gauged supergravity [42–45].

Before discussing the content of the paper, let us consider a pair of caveats. Firstly, it should be kept in mind that the dilatonic deformations are, in general, *not* asymptotically de Sitter. Nonetheless, in the limit of a constant dilaton field, a TdS solution (with flat horizon geometry) will always be obtained [26]. Secondly, it is worth emphasizing that the presence of bulk matter (in this case, the dilaton) will

typically break the conformal symmetry of a dual boundary theory. That is to say, the holographic duality now under consideration can be viewed as a dS analogue [46] of the domain wall–quantum field theory (QFT) correspondence [43,47,44].⁵

The remainder of this paper is organized as follows. In Sec. II, we begin by introducing the relevant action and formulating the $[(n+2)$ -dimensional] bulk solutions of interest: dilatonic deformations of a topological de Sitter spacetime with a flat cosmological horizon [26]. (We will subsequently refer to these as “DTdS” solutions.) We will also (following Ref. [26]) give a brief account of thermodynamic properties and the associated quasi-local stress tensor.⁶

In Sec. III, we consider a (presumably) dual, Euclidean QFT that lives on an asymptotic boundary of the bulk spacetime. In particular, we obtain explicit expressions for the QFT stress tensor and thermodynamics by making appropriate identifications with properties of the DTdS bulk. A Cardy-like form [50] for the QFT entropy is then verified. This result can be viewed as a generalization of the Cardy-Verlinde formula [51,52], as appropriate for a flat horizon geometry [41].⁷ Significantly to later analysis, we also identify the Casimir entropy [51] of the QFT.

The focus of Sec. IV is on prospective (generalized) C functions in a DTdS-QFT holographic framework. Here, we consider a pair of prescriptions for the quantity of interest. The first is based on a formula that expresses the C function in terms of local bulk geometry. (See, for instance, Refs. [34,35].) The second follows from the premise that the Casimir entropy of a boundary theory [51] can be regarded, quite literally, as a Cardy-like “central charge” [50] and, hence, C -function [40]. Both of the prescribed forms are rigorously tested to see if they evolve monotonically with respect to variations in relevant parameters. Given a few justifiable assumptions, we are able to demonstrate that this is, indeed, always the case.

Finally, Sec. V provides a summary and some further discussion.

II. DILATONIC DEFORMATIONS OF TdS SPACETIME

In this section we will begin by formulating the bulk theory of interest; namely, a dilatonic deformation “topological–de Sitter” (DTdS) solution with a flat horizon geometry [26]. Keep in mind that this “DTdS” solution describes a domain wall spacetime with a cosmological (but no black hole) horizon.

A. Domain wall action and solutions

To start off, let us consider an action that describes an $(n+2)$ -dimensional dilaton-gravity theory with a Liouville-like potential. More specifically,

⁵Notably, the domain wall–QFT duality (assuming its validity) includes the AdS-CFT correspondence as a very special case.

⁶For further discussion on generalizing the Brown-York quasi-local formalism [25], see Refs. [48,24,49,23,27].

⁷The Cardy-Verlinde formula has already been generalized for a multitude of holographic scenarios. Consult Refs. [53,54] for a list of relevant citations.

⁴In Ref. [31], however, the author argues against TdS solutions in a dynamical-boundary scenario.

$$I = \frac{1}{16\pi G} \int_{\mathcal{M}} d^{n+2}x \sqrt{-g} \left[R - \frac{1}{2}(\partial\phi)^2 + V_o e^{-a\phi_o} \right] + I_{GH}, \quad (1)$$

where \mathcal{M} represents the manifold, G is the $(n+2)$ -dimensional Newton constant, and where V_o and a are to be regarded as positive constants. Note that we have also included the Gibbons-Hawking surface term, I_{GH} , which is necessary for a well-defined variational principle on the boundary of the manifold [55]. This surface term takes the form

$$I_{GH} = \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^{n+1}x \sqrt{|h|} \mathcal{K}, \quad (2)$$

where \mathcal{K} is the trace of the extrinsic curvature on the boundary $(\partial\mathcal{M})$.

Interestingly, the above action (1) is known to effectively describe a truncated theory of gauged supergravity [42,43]. Given this pedigree, V_o and a can be directly expressed in terms of \mathcal{N} and p , where \mathcal{N} is the number of Dp -branes in the originating theory [44].

A certain class of domain wall–black hole solutions (with Ricci-flat horizons) has been found for this action [45]. The associated metric and dilaton can be expressed by way of the following formalism:

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + R(r)^2 dx_n^2, \quad (3)$$

$$f(r) = \frac{V_o e^{-a\phi_o} r^{2N}}{nN[N(n+2)-1]l^{2N-2}} - \frac{mr^{1-nN}}{\sqrt{2nN(1-N)}}, \quad (4)$$

$$R(r) = \frac{r^N}{l^{N-2}}, \quad (5)$$

$$\phi(r) = \phi_o + \sqrt{2nN(1-N)} \ln(r). \quad (6)$$

In the above, ϕ_o and m are non-negative constants of integration, dx_n^2 is the dimensionless line element of an n -dimensional Ricci-flat spacetime, and l is some length parameter (ensuring correct dimensionality) that will subsequently be set to unity. (Note that $m=0$ corresponds to a purely domain wall spacetime.) Furthermore, the parameter N has been defined in accordance with

$$a = \sqrt{\frac{2nN(1-n)}{nN}}. \quad (7)$$

In view of Eq. (4), the following bound should be imposed on N :

$$(n+2)^{-1} \leq N \leq 1. \quad (8)$$

For later convenience, let us rewrite Eq. (4) in the following form:

$$f(r) = br^{2N} - \frac{\tilde{m}}{r^{nN-1}}, \quad (9)$$

where

$$b \equiv \frac{V_o e^{-a\phi_o}}{nN[N(n+2)-1]}, \quad (10)$$

$$\tilde{m} \equiv \frac{m}{\sqrt{2nN(1-N)}}. \quad (11)$$

It is now quite evident that the special case of $N=1$ describes, with a suitable renormalization of m , a topologically flat Schwarzschild-AdS black hole.

When $N < 1$, this class of solutions is no longer asymptotically AdS. However, one can still obtain a well-defined quasi-local stress tensor [25], provided that an appropriate surface counterterm has first been added to the action [48,24,49,44]. The conserved mass (M) can be directly calculated via this stress-energy tensor, and one finds [45]

$$M = \frac{nN\mathcal{V}_n\tilde{m}}{16\pi G}, \quad (12)$$

where \mathcal{V}_n is the volume of dx_n^2 (i.e., the dimensionless volume of the domain wall). Note that there is no vacuum ($m=0$) contribution to the mass by virtue of the locally flat solution space [49].

Now let us suppose that $V_o < 0$. In this case, the action (1) can still be effectively viewed as the truncation of a gauged supergravity theory [46]. Let us further assume that m (or \tilde{m}) ≤ 0 and then, for the sake of convenience, redefine $V_o \rightarrow -V_o$ and $\tilde{m} \rightarrow -\tilde{m}$. The prior solution remains unchanged except for the metric function $f(r)$, which should now be revised as follows:

$$f(r) = \frac{\tilde{m}}{r^{nN-1}} - br^{2N}. \quad (13)$$

For the special case of $N=1$, the revised solution describes a topological de Sitter spacetime [26] with a flat (cosmological) horizon geometry. For general N (but satisfying Eq. (8)), this dilatonicly deformed solution is no longer asymptotically dS, but it does, in fact, still possess a cosmological horizon. Denoting the horizon location by $r=r_c$, we have

$$r_c = \left[\frac{\tilde{m}}{b} \right]^{1/[N(2+n)-1]}. \quad (14)$$

It should be kept in mind that, for any allowed N , the cosmological horizon encloses a naked singularity.

Ultimately, in the subsequent analysis, we are interested in the region of spacetime outside of the cosmological horizon (i.e., $r > r_c$) and, thus, appropriately shielded from the naked singularity. (In particular, our interest is in the asymptotic limit of $r \rightarrow \infty$; that is, future spacelike infinity or \mathcal{I}_+ [20].) The coordinates r and t change their character in

crossing over the horizon (from spacelike to timelike and vice versa), and so, for illustrative purposes, we will relabel these as $r \rightarrow \tau$ and $t \rightarrow \rho$. Given that $\tau \geq \tau_c = r_c$, the DTdS metric (3),(13) takes on the following suggestive form:

$$ds^2 = -f^{-1}(\tau)d\tau^2 + f(\tau)d\rho^2 + R(\tau)^2 dx_n^2, \quad (15)$$

$$f(\tau) = b\tau^{2N} - \frac{\tilde{m}}{\tau^{nN-1}} \geq 0. \quad (16)$$

For future reference, we also present the geometry of an $(n+1)$ -dimensional spacelike boundary, $\partial\mathcal{M}$:⁸

$$ds_B^2 = h_{ab}dx^a dx^b = f(\tau)d\rho^2 + R(\tau)^2 dx_n^2. \quad (17)$$

Note that for any specific boundary, τ is fixed at some value greater than τ_c .

B. Thermodynamics and stress tensor

The associated thermodynamics of any cosmological horizon can be obtained via standard identifications [56]. For a DTdS solution, in particular, the horizon temperature and entropy are, respectively, calculated as follows:

$$T_H = -\frac{1}{4\pi} \left. \frac{df}{dr} \right|_{r=r_c} = \frac{b[N(n+2)-1]}{4\pi} r_c^{2N-1}, \quad (18)$$

$$S_H = \frac{\text{“area”}}{4G} = \frac{\mathcal{V}_n r_c^{nN}}{4G}. \quad (19)$$

With the above thermodynamic identities, it can readily be shown that the first law of horizon thermodynamics, $dM = T_H dS_H$, is uniquely satisfied (up to the usual constant) for M as given by Eq. (12). However, it is still instructive to derive M on a more fundamental level; namely, as the conserved charge associated with time translation. Notably, such a calculation has already been detailed for the model of current interest [26], so here we will simply quote the pertinent results.

Following a generalized Brown-York treatment [25] (which necessitates that an appropriate surface counterterm be added to the action [24,49,23,27] so as to cancel off any infrared divergences [48]), the authors of Ref. [26] identified

$$T_{ij} = -\delta_{ij} \frac{(2N-1)\tilde{m}\tau^{-(n-1)N}}{16\pi G b^{\frac{1}{2}}}, \quad (20)$$

$$T_{\rho\rho} = \frac{nN\tilde{m}b^{1/2}\tau^{-(n-1)N}}{16\pi G} \quad (21)$$

as the leading-order (in τ^{-1}) components of the quasi-local stress tensor. Keep in mind that our specific interest is in the regime of asymptotically large τ .

⁸Note that Greek indices will imply coordinates of the $(n+2)$ -dimensional manifold and Roman indices will imply boundary coordinates.

Generally speaking, one can apply the quasi-local stress tensor to calculate a conserved charge that is associated with a global symmetry of the spacetime. However, even in the absence of a globally timelike Killing vector, it has been shown that the conserved mass of an asymptotically dS spacetime is still within calculational reach [23]. (What is actually required is a process of analytic continuation.) The authors of Ref. [26] further generalized this prescription to a DTdS spacetime and found the following:

$$M = \mathcal{V}_n \tau^{nN} f^{-1/2}(\tau) T_{\rho\rho} \quad (22)$$

for the conserved mass.

Substituting Eq. (21) into the above relation and taking the asymptotic limit, we find the anticipated result [26]:

$$\lim_{\tau \rightarrow \infty} M = \frac{nN}{16\pi G} \mathcal{V}_n \tilde{m}. \quad (23)$$

Fortunately, this outcome agrees with the priorly quoted result (12) and, thus, with the first law of DTdS horizon thermodynamics.

III. EUCLIDEAN QFTs ON THE BOUNDARY

It has been suggested that the AdS-CFT correspondence [15–17] is really just a special case of a more general holographic duality; namely, the domain wall–QFT correspondence [43,47,44]. By way of analogy, one might argue that the dS-CFT correspondence [14] is also a special case of a more encompassing duality. That is to say, there may exist a dual relationship between certain domain wall spacetimes and Euclidean QFTs [46,26]. Although somewhat speculative, we will adopt this viewpoint for the duration of the paper.

A. QFT geometry and stress tensor

With the above discussion in mind, let us now consider a Euclidean QFT that lives on an asymptotic boundary of a DTdS bulk spacetime. Presumably (or perhaps naively), the bulk and boundary theories should be dually related in a holographic sense. Keep in mind that, for the very special case of $N=1$, the (TdS) bulk theory is asymptotically dS, the boundary theory is a conformal one, and a holographic duality appears to be in evidence [29–31,2].

As is typically the case in holographic bulk-boundary dualities, the metric of the QFT in question will be fixed, up to a conformal factor, as the metric on an asymptotic boundary of the bulk spacetime. By analogy with Ref. [52], we invoke

$$\begin{aligned} ds_{QFT}^2 &= \gamma_{ab} dx^a dx^b \\ &= \lim_{\tau \rightarrow \infty} \frac{1}{\tau^{2N} b} ds^2 \\ &= d\rho^2 + b^{-1} dx_n^2, \end{aligned} \quad (24)$$

where ds^2 (in the second line) is the metric defined by Eqs. (15),(16). Let us reemphasize that all current and/or future considerations are restricted to the region outside of the cosmological horizon (i.e., $\tau \geq \tau_c$).

We can calculate the stress tensor (\mathcal{T}_{ab}) of the QFT by way of the following relation [57]:

$$\sqrt{\gamma}\gamma^{ab}\mathcal{T}_{bc} = \lim_{\tau \rightarrow \infty} \sqrt{h}h^{ab}T_{bc}, \quad (25)$$

where T_{ab} is the quasi-local stress tensor of Eqs. (20),(21) and the boundary metric, h_{ab} , is defined by Eq. (17).

Utilizing the above relation, we are able to deduce

$$\mathcal{T}_{ij} = -\delta_{ij} \frac{(2N-1)\tilde{m}b^{(n-2)/2}}{16\pi G}, \quad (26)$$

$$\mathcal{T}_{\rho\rho} = \frac{nN\tilde{m}b^{n/2}}{16\pi G}. \quad (27)$$

It is interesting to note that the trace of the stress tensor, $\mathcal{T} = \mathcal{T}_a^a$, only vanishes in the special case of $N=1$. Reassuringly, this special case describes an asymptotically dS space-time, and so the corresponding QFT should, indeed, be a conformal theory. Of further interest, the above results imply the following equation of state for the QFT (with ϵ and p , respectively, denoting energy density and pressure):

$$\begin{aligned} \omega \equiv \frac{p}{\epsilon} &= -b \frac{\mathcal{T}_{ij}\delta_j^i}{\mathcal{T}_{\rho\rho}} \\ &= \frac{2N-1}{nN}, \end{aligned} \quad (28)$$

where we have assumed a perfect-fluid description. As expected, this equation reduces to $\omega=1/n$ (i.e., radiative matter) when the $N=1$ conformal theory is realized.

B. QFT thermodynamics

Let us now consider the thermodynamic properties of this QFT and subsequently determine if they can accommodate a Cardy-Verlinde-like entropic form [50,51]. As is the usual practice, we first rescale the QFT metric so that the associated boundary is located at a fixed “radial distance,” \mathcal{R} , from the origin.⁹ This necessitates the following conformal transformation:

$$ds_{QFT}^2 \rightarrow b\mathcal{R}^2 ds_{QFT}^2 = b\mathcal{R}^2 d\rho^2 + \mathcal{R}^2 dx_n^2. \quad (29)$$

However, we have not yet realized the desired form; that is

$$ds_{QFT}^2 = d\rho^2 + \mathcal{R}^2 dx_n^2. \quad (30)$$

Evidently, the Euclidean time coordinate should be further rescaled such that $\rho \rightarrow b^{1/2}\mathcal{R}\rho$. It follows that, from a QFT

perspective, the bulk energy and temperature (having units of inverse time) should be “red shifted” by a factor of $\Delta = [b^{1/2}\mathcal{R}]^{-1}$.

In accordance with the above discussion, the QFT thermodynamics can be identified as follows:¹⁰

$$\begin{aligned} E_{QFT} = \Delta M &= \frac{nN\mathcal{V}_n\tilde{m}}{16\pi G b^{1/2}\mathcal{R}} \\ &= \frac{nN\mathcal{V}_n b^{1/2}}{16\pi G \mathcal{R}} \tau_c^{N(2+n)-1}, \end{aligned} \quad (31)$$

$$T_{QFT} = \Delta T_H = \frac{b^{1/2}[N(n+2)-1]}{4\pi \mathcal{R}} \tau_c^{2N-1}, \quad (32)$$

$$S_{QFT} = S_H = \frac{\mathcal{V}_n}{4G} \tau_c^{nN}, \quad (33)$$

where we have incorporated Eqs. (14),(18),(19),(23). It can readily be verified that these relations satisfy the first law of QFT thermodynamics; that is, $dE_{QFT} = T_{QFT} dS_{QFT}$. We also take note of $E_{QFT} \geq 0$, which is indicative of a topological dS, rather than a Schwarzschild dS, holographic framework [32,29].

In general and regardless of dimensionality, the entropy of any horizon and, by duality, the entropy of its holographic boundary theory should be expressible in a Cardy-like form [58,59]. That is to say, one might expect [50]

$$S_{QFT} = \frac{2\pi}{n} \sqrt{\frac{c}{6} \left[L_o - \frac{c}{24} \right]}, \quad (34)$$

where $L_o = \mathcal{R}E_{QFT}$ and the “central charge,” c , is directly proportional to the Casimir (i.e., sub-extensive) energy of the boundary theory [51]. However, given a bulk theory with a flat horizon geometry (which implies a vanishing Casimir energy), the associated QFT can be expected to conform with the following version [41,54]:

$$S_{QFT} = \frac{2\pi}{n} \sqrt{\frac{cL_o}{6}}. \quad (35)$$

As a consequence of this form, c is, in principle, proportional to an appropriately defined Casimir entropy [51] (which, unlike the Casimir energy, remains finite and positive, regardless of the horizon geometry [41,60]).

We find that the above thermodynamic relations do indeed satisfy Eq. (35) (with $L_o = \mathcal{R}E_{QFT}$), as long as

$$c = \frac{3n\mathcal{V}_n}{2\pi G N b^{1/2}} \tau_c^{N(n-2)+1}. \quad (36)$$

⁹Strictly speaking, \mathcal{R} represents temporal evolution when outside of the cosmological horizon.

¹⁰Note that the entropy is universally unaffected by any such coordinate rescaling [17].

The expected relation between the Casimir entropy (S_C) and the generalized central charge is $S_C = \pi c/6n$ (for instance, [41]). On this basis

$$S_C = \frac{\mathcal{V}_n}{4GNb^{1/2}} \tau_c^{N(n-2)+1}. \quad (37)$$

It is hard to confirm the validity of this result, insofar as we are unable to calculate the Casimir entropy by more direct means. (This is contrary to the usual scenario for a spherical horizon: the Casimir entropy is directly proportional to the Casimir energy, which represents a violation in the Euler identity [51].) Nonetheless, it is quite reassuring that S_C (as defined above) reduces to its anticipated form in the special $N=1$ case [30].

IV. GENERALIZED C FUNCTIONS

With inspiration from the ultraviolet-infrared correspondence [38,39], it is commonly believed that evolution of a bulk spacetime will give rise to some form of RG flow in its holographically related boundary theory (for instance, [33–37,23]). Moreover, there should exist some generalized C function (in analogy with two-dimensional RG flows [40]) that exhibits appropriate monotonicity properties as the state of the system varies. On the basis of such arguments, the existence (or lack thereof) of a suitable C function should serve as an appropriate litmus test for a conjectured bulk-boundary duality. This philosophy, in a DTdS/QFT context, will serve as the premise for the analysis that follows.

Given a bulk spacetime and its dually related boundary theory, there are two commonly used prescriptions for the generalized C function. We will examine both of these in turn.

A. Bulk-geometrical prescription

Firstly, let us consider any bulk $[(n+2)\text{-dimensional}]$ spacetime for which the metric can be expressed in the following domain wall-like form:

$$ds^2 = -dz^2 + e^{2A(z)}[dy^2 + dx_n^2]. \quad (38)$$

In this case, one expects the existence of a generalized C function that is based on local bulk geometry and can be represented as follows (for instance, [34,35]):¹¹

$$C \sim \frac{1}{G[A'(z)]^n}. \quad (39)$$

Note that a prime now indicates differentiation with respect to z .

¹¹A very recent paper [11] considered a revised form for this C function, which apparently has a wider range of applicability. However, in the case of a flat horizon geometry (as is relevant to the current study), this newer formulation reduces to Eq. (39).

The DTdS bulk metric of Eq. (15) can be cast into the above template (38) by way of the following identifications:

$$dz = \frac{1}{\sqrt{f(\tau)}} d\tau, \quad (40)$$

$$y = \frac{\sqrt{f(\tau)}}{R(\tau)} \rho, \quad (41)$$

$$A(z) = A(\tau) = \ln[R(\tau)] = N \ln(\tau). \quad (42)$$

As it stands, such a calculation of C would not be particularly enlightening. Nevertheless, we can still viably proceed by first assuming that the effective mass parameter, \tilde{m} , is much smaller than the other relevant scales. (However, \tilde{m} should remain a non-vanishing quantity, so that a cosmological horizon is still in existence.) It is significant that, with this assumption, $f(\tau) \approx b\tau^{2N}$ becomes a valid approximation [cf. Eq. (16)].

Applying the pertinent approximation to Eq. (40), we obtain the following useful relation:

$$dz \approx \frac{\tau^{-N}}{b^{1/2}} d\tau. \quad (43)$$

In terms of the above formalism, the prescribed C function (39) now yields

$$C \sim \frac{\tau^{n(1-N)}}{GN^n b^{n/2}}. \quad (44)$$

It is immediately clear that C increases monotonically with increasing “radial distance” τ (recalling that $N \leq 1$). Hence, we have confirmed the anticipated ultraviolet-infrared duality [38,39]. That is, the infrared (large τ) limit of the bulk theory corresponds to the ultraviolet (large C) limit of the QFT and *vice versa*.

Interestingly, we see that C becomes a constant (with respect to variations in τ) when $N=1$. That is, the constant-dilaton (TdS) theory translates to a conformal fixed point of the holographic RG flow. Clearly, this $N=1$ fixed point is an infrared one.

It should also be instructive to examine the behavior of the C function under variations in N . Significantly, changes in N reflect variations in the matter content of the theory. To put it another way, as N monotonically decreases below its conformal value of 1, the bulk scalar fields are effectively being “turned on” [cf. Eq. (6)]. In fact, simple analysis tells us that the dilaton field will continue to “grow” until $N=1/2$ has been reached. Thus, one might expect $N=1/2$ to represent a fixed point in the associated RG flow. (Clearly, the conformal value, $N=1$, serves as the other fixed point.) We will provide further support for this claim in the latter part of this section.

To proceed along the suggested line, it is necessary to reexpress Eq. (44) so that all N dependence is explicit. We can accomplish this task by substituting for $b=b(N)$ via Eq. (10). This process yields

$$C \sim \frac{\tau^{n(1-N)}}{G} \left[\frac{nN(n+2)-n}{NV_o} \right]^{n/2}, \quad (45)$$

where we have set $\phi_o = 0$ for the sake of convenience.

Given that C is strictly a positive quantity, it is just as informative (and substantially easier) to consider variations in $\ln(C)$. Hence, it is useful to write

$$\begin{aligned} \ln(C) = & n(1-N)\ln(\tau) + \frac{n}{2}\ln[N(n+2)-1] \\ & - \frac{n}{2}\ln(N) + \text{const.} \end{aligned} \quad (46)$$

Varying this expression with respect to N , we have

$$\frac{\partial \ln(C)}{\partial N} = -\ln(\tau) + \frac{n}{2N}[N(n+2)-1]^{-1}. \quad (47)$$

The above result indicates that, for sufficiently large values of τ , C is a monotonically decreasing function of N . Furthermore, since 1 is an upper bound on N , the conformal theory can be identified as the infrared fixed point with respect to bulk-matter evolution. Given that the $N=1$ limit corresponds to an essentially matter-free theory, this identification seems pleasantly intuitive. At this juncture, however, the ultraviolet fixed point seems somewhat less clear.

Before proceeding on to the next phase of the analysis, let us comment on the condition of “sufficiently large τ .” This constraint can be viewed as a manifestation of a certain aspect of the DTdS-QFT framework. In particular, any external observer will be unable to access information from behind the cosmological horizon and the duality must, therefore, naturally break down when $\tau \leq \tau_c$. To put it another way, τ_c can be viewed as a necessary ultraviolet cutoff for the DTdS bulk or (by duality [38,39]) an infrared cutoff for the QFT.

B. Casimir-entropic prescription

Alternatively, given a holographic boundary theory, the Casimir entropy (S_C) has also been interpreted as a generalized C function. (With considerable success; see, for instance, Refs. [41,61].) This interpretation of S_C follows directly from its role as an effective central charge [50,40] in the Cardy-Verlinde formula [51]. Recalling Sec. III, we have already identified the Casimir entropy (37) for the QFT of interest. On this basis, let us now consider

$$C = S_C = \frac{\mathcal{V}_n}{4GNb^{1/2}} \tau_c^{N(n-2)+1}. \quad (48)$$

Considering our “game plan,” this C function can most conveniently be expressed as an explicit function of T_{QFT} and N . Applying Eq. (10) for $b=b(N)$ and Eq. (32) for the temperature, we eventually obtain the following expression:

$$\begin{aligned} C = & \frac{\mathcal{V}_n}{4G} \left[\frac{n}{V_o} \right]^{nN/2(2N-1)} \\ & \times \left[\frac{N(n+2)-1}{N} \right]^{[N(4-n)-2]/2(2N-1)} \\ & \times [4\pi\mathcal{R}T]^{[N(n-2)+1]/(2N-1)}. \end{aligned} \quad (49)$$

Note that ϕ_o has again been set to vanish and the subscript “QFT” has been dropped from the temperature.

As discussed before, it is both convenient and sufficient to consider the logarithm of C . Up to some irrelevant constant terms, we find the following:

$$\begin{aligned} \ln(C) = & -\frac{nN}{2(2N-1)} \ln\left(\frac{V_o}{n}\right) \\ & + \frac{N(4-n)-2}{2(2N-1)} \ln\left[\frac{N(n+2)-1}{N}\right] \\ & + \frac{N(n-2)+1}{2N-1} \ln(4\pi\mathcal{R}T). \end{aligned} \quad (50)$$

Let us first consider varying $\ln(C)$ with respect to the boundary radius (\mathcal{R}):

$$\mathcal{R} \frac{\partial \ln(C)}{\partial \mathcal{R}} = \frac{N(n-2)+1}{2N-1}. \quad (51)$$

If we assume that (i) $n \geq 2 - (1/N)$ and (ii) $N \geq \frac{1}{2}$, then the following bound can be established:

$$\frac{\partial \ln(C)}{\partial \mathcal{R}} \geq 0. \quad (52)$$

That is, the ultraviolet-infrared connection has (once again) been verified. Some commentary on the assumed conditions is, however, still in order.

Condition (i) simply limits considerations to a bulk theory of dimensionality three or greater, which obviously covers all physically relevant dimensionalities.

Condition (ii) is more difficult to interpret, given that, strictly speaking, N can take on values as low as $(n+2)^{-1}$ [cf. Eq. (8)]. Nonetheless, it is also of relevance that any value of N below $\frac{1}{2}$ translates into a QFT with a negative pressure; cf. Eq. (28). Interestingly, it has been argued that the domain wall–QFT correspondence will break down for these types of negative-pressure states [44,45]. (This argument follows from an observation that gravity fails to decouple from the QFT when the pressure falls below zero.) In view of this consideration, $N \geq 1/2$ seems to be quite a natural constraint. Furthermore, this lower bound on N supports a prior hypothesis; namely, that $N=1/2$ is the most suitable candidate for an ultraviolet fixed point. To reiterate, this hypothesis is based on the observance that, as N decreases below 1, the dilaton will grow until $N=1/2$ has been reached.

Next, let us consider how the C function evolves when the temperature is varied. It is clear that C has the same func-

tional dependence on T as it has on \mathcal{R} . Hence, imposing the same justifiable constraints as before, we have

$$\frac{\partial \ln(C)}{\partial T} \geq 0. \quad (53)$$

That is, the C function evolves monotonically with respect to temperature and, moreover, the QFT flows towards the ultraviolet as temperature increases. This outcome agrees with the usual expectation that thermal excitations will induce additional degrees of freedom.

Finally, let us consider variations in the C function with respect to the parameter N ; keeping in mind that a decreasing N translates into excitations of the bulk scalar. By way of Eq. (50), the following is found:

$$\begin{aligned} \frac{\partial \ln(C)}{\partial N} = & -\frac{n}{(2N-1)^2} \left[\ln(4\pi\mathcal{R}T) - \frac{1}{2} \ln \left(\frac{N(n+2)-1}{nNV_o^{-1}} \right) \right] \\ & - \frac{N(n-4)+2}{2N(2N-1)} [N(n+2)-1]^{-1}. \end{aligned} \quad (54)$$

It is evident that, for both “sufficiently large” values of temperature and $N \geq \frac{1}{2}$, the following relation will always be satisfied:

$$\frac{\partial \ln(C)}{\partial [-N]} \geq 0. \quad (55)$$

Hence, under suitable conditions, C is a monotonically increasing function as N decreases. This result agrees with our previous finding; thus reconfirming the intuitive notion of bulk matter fields inducing a flow to the ultraviolet.

Let us now comment on the most recently imposed conditions. $N \geq \frac{1}{2}$ is just the previously discussed positive-pressure constraint, whereas the condition of large temperature can be justified by virtue of the following argument. The Cardy-Verlinde formula, upon which this definition of C has been based, only has validity in a regime of large temperature [51]. In fact, a breakdown can be expected in the Cardy-Verlinde formalism when $\mathcal{R}T \gg 1$ is no longer satisfied [62–64]. It is clear that the same limitation can be deduced from our findings.

As an aside, it would be interesting to determine if there is some finite temperature at which Eq. (55) does indeed begin to fail. Significantly, this special value of temperature could be interpreted as the analogue of the Hawking-Page (Schwarzschild-AdS) phase transition [65]. However, it appears that such a determination would require a complicated numerical analysis.

To briefly summarize, we have demonstrated that both definitions of the generalized C function (39), (48) satisfy the monotonicity properties that would be expected for a bulk spacetime with a QFT dual.

V. CONCLUDING DISCUSSION

In summary, we have been investigating the holographic properties of a special class of domain wall solutions: those

having the distinction of a singularity enclosed by a cosmological horizon. Any of these bulk solutions can be interpreted as a dilatonic deformation of a topological de Sitter spacetime [26]. That is to say, in the limit of a constant dilaton field, the once-deformed model will describe an asymptotically de Sitter solution with a cosmological singularity. As discussed in Sec. I, such a singularity may not be a particularly critical issue from the perspective of an asymptotic boundary observer [27]. Also of note, any of these deformed “DTdS” solutions can be analytically continued into a domain wall spacetime that effectively describes a truncated theory of gauged supergravity [42–44].

The initial phase of the analysis entailed strictly bulk considerations. We began here by introducing the relevant (arbitrary-dimensional) action, which describes gravity coupled to a dilaton field with a Liouville-like potential. For this action, a certain class of domain wall–black hole solutions are known [45], whereby a trivial redefinition of the potential leads to a DTdS solution space. We also discussed the thermodynamics and quasi-local stress tensor [25] of these bulk solutions (which were first analyzed in Ref. [26]). Most significantly, the conserved mass is precisely that which satisfies the first law of cosmological horizon thermodynamics.

In the second phase of the analysis, we considered a Euclidean quantum field theory that lives on an asymptotic boundary of the DTdS bulk spacetime. It was argued that the bulk and boundary theories could well have a dual relationship [46] in analogy with the domain wall–QFT correspondence [43,47,44]. Utilizing standard holographic relations [57,51,52], we were able to identify the stress tensor and thermodynamic properties of this QFT. It was then shown that the QFT entropy satisfies a generalized form of the Cardy-Verlinde formula [50,51]. Notably, this generalization can be viewed as the appropriate one for a flat horizon geometry [41,54]. On the basis of this formulation, we also identified the Casimir (i.e., sub-extensive) entropy of the QFT.

The final phase of the analysis focused on generalized C functions. In this regard, we studied two commonly used prescriptions: (i) a formula that expresses C in terms of local bulk geometry (for instance, [34,35]) and (ii) an identification between C and a QFT-induced Casimir entropy (for instance [41,61]). After formulating the appropriate expressions, we tested these prescriptions by varying each of the C functions with respect to relevant parameters. For both versions, the ultraviolet-infrared correspondence [38,39] was clearly established. (That is, C increases monotonically with respect to an increasing boundary radius.) Furthermore, after imposing a few justifiable conditions, we were able to show that C increases monotonically with respect to both increasing temperature and decreasing N ; the latter being a parameter of the bulk theory.¹² Both of these outcomes agree with prior expectations, given that thermal excitations should ac-

¹²For the sake of accuracy, let us note that the temperature correspondence was only verified with the second prescription, whereas the N correspondence was verified for both.

tivate degrees of freedom and a decrease in N can be correlated with the activation of bulk matter fields.

Also of interest, a pair of fixed points was identified for the implied renormalization group flows. In particular, the conformal (or constant dilaton) case of $N=1$ corresponds to an infrared fixed point, while $N=1/2$ describes an ultraviolet fixed point. The significance of the ultraviolet limit is that it describes a boundary theory with vanishing pressure; at which point a breakdown can be expected in the (prospective) DTdS-QFT duality [44,45].

In conclusion, the results of our analysis are definitely in support of a DTdS-QFT and (hence) TdS-CFT correspondence. Further support for the latter duality has come in prior studies; most notably, it has been observed that a TdS bulk gives rise to a positive-energy CFT [29,2] (in direct contrast to the more conventional Schwarzschild-dS bulk scenario [32]). However, we do not mean to imply that these studies

in any way verify the legitimacy of topological de Sitter solutions (i.e., asymptotically dS solutions with a cosmological singularity). Rather, we view these positive outcomes as an argument that such solutions should not be disregarded *a priori*. Let us remind the reader that an asymptotic boundary observer would be causally inhibited from accessing any information from behind the TdS cosmological horizon, including information about the naked singularity [27]. Hence, the potential legitimacy of TdS solutions seems to depend on what constitutes the “fundamental” theory; the bulk or the boundary. It appears that this issue necessitates further investigation.

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